The Convolution Theorem

The Convolution Theorem states:

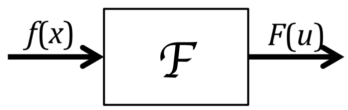
$$f(x) * g(x) \quad <=> F(u)G(u) \quad and \;\; f(x)g(x) \quad <=> F(u) * \; G(u)$$

Proof:

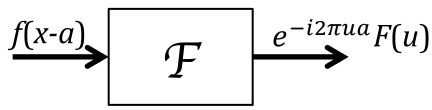
Part I: Proof of the Shift Theorem or shift-invariance:

First we prove the ***shift property*,** that is, that the Fourier Transform ($\mathcal{F}$), as a "system," is shift-invariant:

That is, if



then



That means that adding a shift (x-a) doesn't change the spectrum, $F(u)$, it just adds a *linear phase*, which amounts to multiplication by $e^{-i2\pi ua}$. Multiplying $F(u)$ by $e^{-i2\pi ua}$ for different a translates or shifts $f(x)$ by a.

Let $x' = x-a$ and $dx = dx'$

Then $F[f(x-a)] = F[f(x')] = \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi[u(x'+a)]} dx'$

Now $e^{-i2\pi u(x'+a)} = \; e^{-i2\pi ua} \;\; e^{-i2\pi ux'} \;\; where \; e^{-i2\pi ua} \;\; is\; a\; constant$:

So $F[f(x-a)] = e^{-i2\pi ua}\, F(u)$ which is the "**Shift Theorem**"

Therefore, the Fourier Transform, as a ***System***, is "**shift-invariant**."

Significance: Shifting or translating a spatial function a distance $x = a$ adds a linear phase $\theta = ua$ to the original phase. Conversely, a linear phase filter produces a translation of the image. The magnitude spectrum remains unchanged.

Part II: Proof of the Convolution Theorem:

By definition, $f(x) * g(x) = \int_{-\infty}^{\infty} f(x)\ g(y-x)dx$

The Fourier Transform of $f(x) * g(x)$ *is*

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x)\ g(y-x)\, dx\right] e^{-i2\pi uy}\, dy$$

That is, after the convolution integral is evaluated, the [ ] brackets will contain a function of y, so the variable of integration for the Fourier Transform is y.

Thinking of these nested integrals as a double summation loop in x and y, with y being the outside loop:

$\sum_{y=-\infty}^{\infty} \sum_{x=-\infty}^{\infty} \ldots$ we can easily reverse the order of integration $\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \ldots$ to

$$\int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} g(y-x)\ e^{-i2\pi uy}\, dy\right] dx$$

- reversing the order of integration allows the [ ] brackets to be moved to include the terms over which we are integrating
- $f(x)$ can be treated as a constant when integrating over y, and can therefore be pulled outside of the inner integral.

But, by the Shift Property (Part I),

$$\int_{-\infty}^{\infty} g(y-x)\ e^{-i2\pi uy}\, dy = F[g(y-x)] = e^{-i2\pi ux}\ G(u)$$

So $\int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} g(y-x)\ e^{-i2\pi uy}\, dy\right] dx = \int_{-\infty}^{\infty} f(x)\ e^{-i2\pi ux}\, G(u) dx$

$$= \left[\int_{-\infty}^{\infty} f(x)\ e^{-i2\pi ux}\, dx\right] G(u)$$

$$= F(u)\, G(u)$$

Since the Fourier Transform $F[f * g] = F(u)\, G(u)$, Convolution in the space/time domain is equivalent to multiplication in the frequency domain.